PART-II (PAPER-III)

ABSTRACT ALGEBRA

GROUPS

(1) Binary Operation :- An operation which combines two elements of a set to produce another elements of the same set is called a binary operation

Generally the symbol "o" or "*" is used to represent a binary operation.

A binary operation in a set S is said to be commutative if $aob = boa$, $\forall a, b \in S$

Associative if

$$ao(boc) = (aob)oc, \ \forall \ a, b, c \in S$$

and distributive over addition if

$$ao(b+c) = aob + aoc$$

$$(b+c)oa = boa + coa, \ \forall \ a, b \in S.$$

Illustration :- (i) Addition in a binary operation in case of integers.

(ii) Multiplication is a binary operation in case of complex numbers.

(2) GROUP :- Definition :- Let G be a non-empty set and O be a binary operation, Then the set G together with the operation "O" satisfying the following postulate is called a "Group"

If $a, b, c \in G$ then

(i) $aob \in G$ (Closure axiom)

(ii) The operation is associative

i.e. $ao(boc) = (aob)oc$ (Associative axiom)

(iii) There exists an element $e \in G$ such that

$$aoe = eoa = a$$

The element e is called an identity element in G.

(IV) For every $a \in G$, there exists an element $a^{-1} \in G$ such that $a^{-1}oa = aoa^{-1} = e$

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$\bar{a}^{1}$ is called the inverse of a in G (Inverse axiom)

(3) Finite and Infinite Group:-

(A) Finite Group:- A group is called a finite group if the number of elements in the group is finite. The number of elements in a group is called its order.

(B) Infinite Group:- If the group contains an infinite number of elements, then the group is called an infinite group. An infinite group has an infinite order.

(4) Commutative or Abelian Group:- A Group (G,o) is called an abelian if

$$aob = boc \ \forall \ o \in G.$$

If the group G does not satisfy commutative axioms $aob = boa$, then G is called non-abelian group

~~(5) General Properties of Groups:.~~

ILLUSTRATIVE EXAMPLE

Example:- Prove that the set of positive integers is not a group under addition.

Ans.. Let $P = \{0,1,2,3 \ldots \ldots\}$

(i) The sum of two positive integers is a positive integer. Hence the first Postulate is satisfied.

(ii) The addition of positive integers is associative, for the addition of integers is associative.

(iii) The identity of P is 0. for $a+0=0=0+a=a$, for every $a \in P$.

(iv) The inverse of $a \in P$ is $-a$ which is a negative integer and it does not belong to P.

Thus the fourth Postulate is not satisfied. and hence P is not a group.

Example (2) Prove that the set of even integers forms an Abelian group under addition.

Soln :- Let E be the set of even integers, that is $E=\{\ldots -4, -2, 0, 2, 4, 6 \ldots\}$

(i) The sum of two even integers is an even integer. Hence if $a, b \in E$ then $a+b \in E$

(ii) The addition of integers is associative and hence the addition of even integers is associative.

(iii) The Identity of E is 0. for $a+0=0+a=a$ for all $a \in E$

(iv) The inverse of an even integers is an even integer and hence the inverse of $a \in E$ is $-a \in E$, for $a+(-a)=(-a+a)=0$

Thus all the group postulates are satisfied and hence the set E is a group. Moreover E is an Abelian since the addition in E is commutative, because addition is commutative in the set of integers.

Example (3) Prove that the set of rational numbers is an Abelian group under addition.

Ans :- Let Q be the set rational numbers, that is numbers of the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$. If the set Q is an Abelian group under addition, then it must satisfy all the five conditions

(i) If $\frac{a}{b}$ and $\frac{c}{d} \in Q$ then $\frac{a}{b}+\frac{c}{d}$ which is $=\frac{ad+bc}{bd}$ also $\in Q$ Thus Condition (i) is satisfied

(ii) If $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in Q$ then $\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f}=\frac{ad+bc}{bd}+\frac{e}{f}$

$=\frac{adf+bcf+ebd}{bdf}$

(4)

Also, $\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)=\frac{a}{b}+\frac{cf+ed}{df}=\frac{adf+bcf+bed}{bdf}$

$\therefore \left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f}=\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)$

Thus condition (ii) is satisfied.

(iii) The identity is zero, for $\frac{a}{b}+0=\frac{a}{b}$

(iv) The inverse of $\frac{a}{b}$ is $\left(-\frac{a}{b}\right)$ for $\frac{a}{b}+\left(-\frac{a}{b}\right)=0$.

(v) $\frac{a}{b}+\frac{c}{d}=\frac{c}{d}+\frac{a}{b}$.

Thus we see that the set Q satisfies all the five conditions of a group under addition and hence it is an Abelian group w.r.to addition.

Q Show that the set of all real numbers except -1 forms a group under binary operation $*$ defined by $a*b=a+b+ab$.

Ans:- It has to be noted that for any $a,b \in S$, $a+b+ab \neq -1$ otherwise

$a+b+ab=-1 \Rightarrow (1+a)b=-(1+a)$

$\Rightarrow (1+a)(1+b)=0 \Rightarrow a=-1$ or $b=-1$

Since $a\neq -1$ $\therefore$ we shall have $b=-1$

which contradict the definition of S

Thus for any $a,b \in S$, $a+b+ab \neq -1$

(i) Obviously $a*b=a+b+ab$ is a real number different from -1. Hence S is closed w.r.t. $*$

(ii) Let $a,b,c \in S$.

Then $a*(b*c)=a*(b+c+bc)=a+(b+c+bc)+a(b+c+bc)$

$=a+b+c+bc+ab+ac+abc=a+b+c+ab+bc+ca+abc$

Similarly $(a*b)*c = a+b+c+ab+bc+ca+abc$

$\therefore a*(b*c) = (a*b)*c$

Hence $*$ is associative

(iii) The number $0 \in S$ is the identity element

For $a*0 = a+0+a\cdot 0 = a+0+0 = a$

$0*a = 0+a+0a = 0+a+0 = a \quad \forall\ a \in S$

(iv) In order that an element of $a \in S$ may be the multiplicative inverse of $a \in S$, we must have

$a*a' = a+a'+a'a = 0$

$\Rightarrow a+(1+a)a' = 0 \Rightarrow a' = -\dfrac{a}{1+a}$ since $1+a \neq -1$

Thus the element $-\dfrac{a}{1+a}$ is the multiplicative inverse of $a \in S$

Hence $(S, *)$ is a group.

(v) Moreover the group is Abelian, since

$a*b = a+b+ab = b+a+ba = b*a$ for all $b \in S$

Q Prove that the nth roots of unity forms a multiplication group.

Ans- Let S be the set of nth roots of unity, that is

$$S = \{ e^{\frac{2r\pi i}{n}},\ r = 0,1,2,3 \ldots (n-1)\},\ \text{where } i = \sqrt{-1}$$

Let a and $b \in S$ given by $a = e$, $r_1 \leq n-1$ and $b = e^{\frac{2r\pi i}{n}}$

$r_2 \leq n-1$, First of all we have to show that $ab \in S$

Now, $ab = e^{\frac{2(r_1+r_2)\pi i}{n}}$

If $r_1+r_2 \leq n-1$ then obviously $ab \in S$

If $r_1+r_2 > n-1$ then let $r_1+r_2 = n+k$ so that $k \leq n-2$

In this case, $ab = e^{\frac{2(n+k)\pi i}{n}} = e^{2\pi i} \cdot e^{\frac{2k\pi i}{n}} = e^{\frac{2k\pi i}{n}} \in S$

Since $k \leq n-2$

Thus in both the cases $ab \in S$.

Hence the first Postulate is satisfied.

The elements of S are complex numbers and the multiplication of complex numbers is associative.

Hence the Second Postulate is satisfied.

The identity of S is $e^{\frac{2 \cdot 0 \cdot \pi i}{n}} = 1$ for $r=0$

The inverse of any element $e^{\frac{2r\pi i}{n}}$ is $= e^{\frac{2(n-r)\pi i}{n}}$

for their Product $= e^{\frac{2n\pi i}{n}} = e^{2\pi i} = 1$

The third and fourth Postulates are satisfied

Hence the set S forms a group

Moreover the multiplication of two complex numbers is commutative

Hence S is an Abelian group.

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Q Show that the set S of all Positive rationals forms a group under the binary operation $*$ defined by $a * b = \frac{ab}{2}$

Ans:- (i) For any Positive rationals a, b, $\frac{1}{2}ab$ is also a Positive rational. Hence the operation $*$ is binary operation on S.

(ii) Let $a, b, c \in S$.

Then we have $a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{1}{2}\left(a \cdot \frac{bc}{2}\right)$

Similarly $(a*b)*c = \frac{1}{4}abc$, Thus $a*b*c = (a*b)*c$

Hence $*$ is associative on S.

(iii) The identity in S is 2. Since $a*2 = \frac{1}{2}a2 = a$

and $2*a = \frac{1}{2}2a = a$

(iv) The inverse of $a \in S$ is $\frac{4}{a} \in S$,

Since $a*\frac{4}{a} = \frac{1}{2}a\cdot\frac{4}{a} = 2$ and $\frac{4}{a}*a = \frac{1}{2}\cdot\frac{4}{a}a = 2$

Hence $(S,*)$ is a group

(v) Moreover the group is Abelian, Since $a*b = \frac{1}{2}ab = \frac{1}{2}ba = b*a$

for all $a,b \in S$

Q:- Prove that the set of real numbers forms an Abelian group under addition

Ans:- Let R be the set of real numbers

(i) The sum of two real numbers is a real number. Hence if $a, b \in R$ then $a+b \in R$

(ii) The addition of real numbers is associative. Hence if $a,b,c \in R$, then $a+(b+c) = (a+b)+c$

(iii) The identity of R is $0 \in R$, for $a+0 = 0+a = a \; \forall a \in R$

(iv) The inverse of $a \in R$ is $-a \in R$ for $a+(-a) = (-a)+a = 0$

Thus all the group postulates are satisfied and hence R is a group. Moreover R is an Abelian group since addition in R is commutative